

NASH PROBLEM FOR SURFACES

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ABSTRACT. We prove that Nash mapping is bijective for any surface defined over an algebraically closed field of characteristic 0.

1. INTRODUCTION

Nash problem [15] was formulated in the sixties (but published later) in the attempt to understand the relation between the structure of resolution of singularities of an algebraic variety X over a field of characteristic 0 and the space of arcs (germs of parametrized curves) in the variety. He proved that the space of arcs centred at the singular locus (endowed with an infinite-dimensional algebraic variety structure) has finitely many irreducible components and proposed to study the relation of these components with the essential irreducible components of the exceptional set a resolution of singularities.

An irreducible component E_i of the exceptional divisor of a resolution of singularities is called essential, if given any other resolution the birational transform of E_i to the second resolution is an irreducible component of the exceptional divisor. Nash defined a mapping from the set of irreducible components of the space of arcs centred at the singular locus to the set of essential components of a resolution as follows: he assigns to each component W of the space of arcs centred at the singular locus the unique component of the exceptional divisor which meets the lifting of a generic arc of W to the resolution. Nash established the injectivity of this mapping. For the case of surfaces it seemed possible for him that the mapping is also surjective, and posed the problem as an open question. He also proposed to study the mapping in the higher dimensional case. Nash resolved the question positively for the A_k singularities. As a general reference for Nash problem the reader may look at [15] and [5].

Besides Nash problem, the study of arc spaces is interesting because it lays the foundations for motivic integration and because the study of its geometric properties reveals properties of the underlying varieties (see papers of Denef, Loeser, de Fernex, Ein, Ishii, Lazarsfeld, Mustata, Yasuda and others).

It is well known that birational geometry of surfaces is much simpler than in higher dimension. This fact reflects on Nash problem: Ishii and Kollar showed in [5] a 4-dimensional example with non-bijective Nash mapping. In the same paper

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they showed the bijectivity of the Nash mapping for toric singularities of arbitrary dimension. Other advances in the higher dimensional case include [21], [4], [11].

On the other hand bijectivity of the Nash mapping has been shown for many classes of surfaces (see [4],[5],[7],[8],[9],[10],[14],[16],[18],[19],[22], [20],[23],[24]). The techniques leading to the proof of each of these cases are different in nature, and the proofs are often complicated. It is worth to notice that even for the case of the rational double points not solved by Nash a complete proof has to be awaited until last year: see [19], [22] and [16]; in the last paper the result is proved for all quotient surface singularities.

In this paper we resolve Nash question for surfaces:

Main Theorem. *Nash mapping is bijective for any surface defined over an algebraically closed field of characteristic 0.*

The core of the result is the case of normal surface singularities. After settling this case we deduce from it the general surface case following a suggestion by C. Plénat and M. Spivakovsky.

The proof is based on the use of convergent wedges and topological methods. A wedge is a uniparametric family of arcs. The use of wedges in connection to Nash problem was proposed by M. Lejeune-Jalabert [9]. Later A. Reguera [25], building onto the fundamental Lemma of motivic integration by J. Denef and F. Loeser [1], proved a characterization of components which are at the image of the Nash map in terms of formal wedges defined over fields which are of infinite transcendence degree over the base field. In [3] it is proved a characterization of the image of the Nash mapping for surfaces in terms of convergent (or even algebraic) wedges defined over the base field, which is the starting point of this article. In the same paper it is shown that Nash problem is of topological nature. Independently, in [11], it is given a different sufficient condition for being at the image of the Nash map in terms of formal wedges defined over the base field; this condition holds in arbitrary dimension, but it is weaker than the one given in [3] for the surface case. In [16] the second author settles Nash question for quotient surface singularities. The present paper is inspired in the ideas of [16], more concretely in the use of representatives of wedges and in the use of topological methods.

The proof in [16] uncovers a nice phenomenon of the deformation theory of curves in surfaces, which is worth to be studied on its own, and which perhaps could give a different proof for the bijectivity of Nash mapping for surfaces.

The idea of our proof is as follows: let (X, O) be a normal surface singularity and

$$\pi : \tilde{X} \rightarrow (X, O)$$

be the minimal resolution of singularities. By a Theorem of [3] if Nash mapping of (X, O) is not bijective there exists a convergent wedge

$$\alpha : (\mathbb{C}^2, O) \rightarrow (X, O)$$

with certain precise properties. As in [16], taking a suitable representative we may view α as a uniparametric family of mappings

$$\alpha_s : \mathcal{U}_s \rightarrow (X, O)$$

from a family of domains \mathcal{U}_s to X with the property that each \mathcal{U}_s is diffeomorphic to a disc. For any s we consider the lifting

$$\tilde{\alpha}_s : \mathcal{U}_s \rightarrow \tilde{X}$$

to the resolution. Notice that $\tilde{\alpha}_s$ is the normalization mapping of the image curve.

On the other hand, if we denote by Y_s the image of $\tilde{\alpha}_s$ for $s \neq 0$, then we may consider the limit divisor Y_0 in \tilde{X} when s approaches 0. This limit divisor consists of the union of the image of $\tilde{\alpha}_0$ and certain components of the exceptional divisor of the resolution whose multiplicities are easy to be computed. We prove an upper bound for the Euler characteristic of the normalization of any reduced deformation of Y_0 in terms of the following data: the topology of Y_0 , the multiplicities of its components and the set of intersection points of Y_0 with the generic member Y_s of the deformation. Using this bound we show that the Euler characteristic of the normalization of Y_s is strictly smaller than one. This contradicts the fact that the normalization is a disc.

In the last Section we deduce the general case from the normal case following a remark by C. Plénat and M. Spivakovsky.

2. PRELIMINARIES

2.1. Let (X, O) be a complex analytic normal surface singularity. Let

$$\pi : (\tilde{X}, E) \rightarrow (X, O)$$

be the minimal resolution of singularities, which is an isomorphism outside the exceptional divisor $E := \pi^{-1}(O)$. Consider the decomposition $E = \cup_{i=0}^r E_i$ of E into irreducible components. These irreducible components are the essential components of (X, O) .

The germ (X, O) is embedded in an ambient space \mathbb{C}^N . Denote by B_ϵ the closed ball of radius ϵ centred at the origin and by \mathbb{S}_ϵ its boundary sphere. Take a Milnor radius ϵ_0 for (X, O) in \mathbb{C}^N (in particular $X \cap B_{\epsilon_0}$ has conical differentiable structure). From now on we will denote by X a representative $X \cap B_{\epsilon_0}$ of X and by \tilde{X} the resolution of singularities $\pi^{-1}(X)$. In these conditions the space \tilde{X} admits the exceptional divisor E as a deformation retract. Hence the homology group $H_2(\tilde{X}, \mathbb{Z})$ is free and generated by the classes of the irreducible components E_i . Since \tilde{X} is a smooth 4-manifold there is a symmetric intersection product

$$\cdot : H_2(\tilde{X}, \mathbb{Z}) \times H_2(\tilde{X}, \mathbb{Z}) \rightarrow \mathbb{Z}.$$

The intersection product is negative definite since it is the intersection product of a resolution of a surface singularity.

2.2. We recall some terminology and results from [3]. Consider coordinates (t, s) in the germ (\mathbb{C}^2, O) . A *convergent wedge* is a complex analytic germ

$$\alpha : (\mathbb{C}^2, O) \rightarrow (X, O)$$

which sends the line $V(t)$ to the origin O . Given a wedge α and a parameter value s , the arc

$$\alpha_s : (\mathbb{C}, 0) \rightarrow (X, O)$$

is defined by $\alpha_s(t) = \alpha(t, s)$. The arc α_0 is called *the special arc* of the wedge. For small enough $s \neq 0$ the arcs α_s are called *generic arcs* of α .

Any arc

$$\gamma : (\mathbb{C}, 0) \rightarrow (X, O)$$

admits a unique lifting $\tilde{\gamma}$ to (\tilde{X}, O) .

Definition 1 ([3]). *A convergent wedge α realizes an adjacency from E_j to E_i (with $j \neq i$) if and only if the lifting $\tilde{\alpha}_0$ of the special arc meets E_i transversely at a non-singular point of E and the lifting $\tilde{\alpha}_s$ of a generic arc satisfies $\tilde{\alpha}_s(0) \in E_j$.*

Our proof is based in the following Theorem, which is the implication “(1) \Rightarrow (a)” of Corollary B of [3]:

Theorem 2 ([3]). *An essential divisor E_i is in the image of the Nash mapping if there is no other essential divisor $E_j \neq E_i$ such that there exists a convergent wedge realizing an adjacency from E_j to E_i .*

2.3. The previous theorem allows to address Nash question in the complex analytic case. Suppose that (X, O) is a singularity of a normal algebraic surface defined over an algebraically closed field \mathbb{K} of characteristic 0. It is well known that (X, O) may be defined over a field \mathbb{K}_1 which is a finite extension of \mathbb{Q} , and hence admits an embedding into \mathbb{C} . Let $\bar{\mathbb{K}}_1$ the algebraic closure of \mathbb{K} . We have then two field embeddings $\bar{\mathbb{K}}_1 \subset \mathbb{K}$ and $\bar{\mathbb{K}}_1 \subset \mathbb{C}$. Since the bijectivity of the Nash mapping does not change by extension of algebraically closed fields we deduce that if we prove the bijectivity of the Nash mapping for any complex analytic normal surface singularity, then it holds for any normal surface singularity defined over a field of characteristic equal to 0.

2.4. Following [16] we shall work with representatives rather than germs in order to get richer information about the geometry of the possible wedges. Remember that X stands for a Milnor representative $X \cap B_{\epsilon_0}$ for a Milnor radius ϵ_0 for (X, O) .

Given any non-constant arc germ $\gamma : (\mathbb{C}, O) \rightarrow (X, O)$ there exist a suitable representative

$$\gamma : \Omega \rightarrow X,$$

where Ω is an open neighbourhood of the origin of \mathbb{C} , and a positive radius ϵ such that the mapping

$$\gamma : \gamma^{-1}(B_\epsilon) \rightarrow B_\epsilon$$

is proper and transversal to the sphere \mathbb{S}_ρ for any $0 < \rho \leq \epsilon$ and, moreover, the preimage $\gamma^{-1}(B_\epsilon)$ is connected. In this case $\gamma^{-1}(B_\epsilon)$ is diffeomorphic to a closed disc.

Definition 3 ([16]). *A Milnor representative of γ is a representative of the form*

$$\gamma|_D : D \rightarrow X$$

where D is the disc $\gamma^{-1}(B_\epsilon)$ for ϵ as above.

Denote by D_δ the disc of radius δ centred at the origin of \mathbb{C} .

Given a wedge α with non-constant special arc α_0 , we consider the radius ϵ associated to a Milnor representative of α_0 such that α is defined in a neighbourhood of $D \times D_\delta$ for a positive δ . The radius δ can be chosen small enough so that the mapping α_s is transversal to \mathbb{S}_ϵ for any $s \in D_\delta$. We consider the mapping

$$\beta : (\mathbb{C}^2, O) \rightarrow (\mathbb{C}^N \times D_\delta, (O, 0))$$

given by $\beta(t, s) := (\alpha(t, s), s)$ and define $\mathcal{U}_{\epsilon, \delta} := \beta^{-1}(B_\epsilon \times D_\delta)$. We have that the restriction to $\mathcal{U}_{\epsilon, \delta}$ of the natural projection to D_δ is a smooth trivial fibration with fibre diffeomorphic to a closed disc.

Definition 4 ([16]). *A good representative of a wedge α is a representative of the form*

$$\alpha|_{\mathcal{U}_{\epsilon, \delta}} : \mathcal{U}_{\epsilon, \delta} \rightarrow X$$

with $\mathcal{U}_{\epsilon, \delta}$ and δ chosen as in the previous discussion.

From now on we only deal with wedges with non-constant special arc, which, therefore, have good representatives.

Given a good representative $\alpha|_{\mathcal{U}_{\epsilon, \delta}}$, we denote the corresponding $\mathcal{U}_{\epsilon, \delta}$ simply by \mathcal{U} and consider the representative

$$\beta|_{\mathcal{U}} : \mathcal{U} \rightarrow X \times D_\delta.$$

Given any $s \in D_\delta$ we denote by \mathcal{U}_s the fibre by the natural projection of \mathcal{U} onto D_δ ; it is a region in \mathbb{C} diffeomorphic to a disc. The fact that \mathcal{U}_s is a disc is a key in the proof as it was in the final step of the proof of the main result of [16].

3. WEDGES AND DIVISORS

3.1. Given a wedge α realizing an adjacency, consider a good representative $\alpha|_{\mathcal{U}}$ as in Definition 4.

Since $\beta|_{\mathcal{U}}$ is proper, its image $H := \beta(\mathcal{U})$ is a 2-dimensional closed analytic subset of $X \times D_\delta$. For any $s \in D_\delta$ the fibre H_s by the natural projection onto D_δ is the image of the representative

$$\alpha_s|_{\mathcal{U}_s} : \mathcal{U}_s \rightarrow X.$$

Given the minimal resolution of singularities

$$\pi : \tilde{X} \rightarrow X$$

we consider the mapping

$$\sigma : \tilde{X} \times D_\delta \rightarrow X \times D_\delta.$$

The inverse image $\sigma^{-1}(H)$ is a divisor in the smooth 3-fold $\tilde{X} \times D_\delta$, which can be decomposed as

$$\sigma^{-1}(H) = Y + \sum_{i=0}^r n_i (E_i \times D_\delta),$$

where Y denotes the strict transform of H .

For any $s \in D_\delta$ there is a unique lifting

$$(1) \quad \tilde{\alpha}_s : \mathcal{U}_s \rightarrow Y_s \subset \tilde{X}$$

of $\alpha_s|_{\mathcal{U}_s}$ to \tilde{X} .

For $s \neq 0$ the fibre Y_s is reduced and coincides with the image of the mapping $\tilde{\alpha}_s$. Therefore $\tilde{\alpha}_s$ is the normalization mapping of the curve Y_s . For $s = 0$ the divisor $Y_0 \subset X$ decomposes as

$$(2) \quad Y_0 = Z_0 + \sum_{i=0}^r a_i E_i,$$

where Z_0 is the image of the lifting $\tilde{\alpha}_0$.

3.2. If α is a wedge realizing the adjacency from E_j to E_0 with $j \neq 0$, then by definition, the lifting $\tilde{\alpha}_0$ meets E_0 transversely. In particular, $Z_0 \cdot E_0 = 1$ and $Z_0 \cdot E_i = 0$ for $i > 0$, where Z_0 is as in Formula (2).

Since the divisor Y_s is a deformation of the divisor Y_0 we have the equality

$$(3) \quad Y_0 \cdot E_i = Y_s \cdot E_i$$

for any i . Denote by b_i the intersection product of $Y_s \cdot E_i$ and by M the intersection matrix of the intersection product in \tilde{X} . Then, (3) can be expressed as follows:

$$(4) \quad M(a_0, \dots, a_r)^t = (-1 + b_0, b_1, \dots, b_r)^t.$$

In the terminology of [16] the number b_i is the number of *returns* of the wedge *through the divisor* E_i : it is the number of points $p \in \alpha_s|_{\mathcal{U}_s}^{-1}(O)$ for which the lifting to \tilde{X} of the germ at p of $\alpha_s|_{\mathcal{U}_s}$ meets E_i . The use of returns was one of key ideas introduced by the second author in [16].

Since α realizes an adjacency from E_j to E_0 we have more restrictions about b_i 's and a_i 's. They can be seen as consequences of the following lemma:

Lemma 5. *All the entries of the inverse matrix M^{-1} are non-positive.*

Proof. The matrix $-M$ is symmetric, positive definite, and such that any non-diagonal entry is non-positive. Hence, if endow \mathbb{R}^r with the standard euclidean product then there is a basis v_1, \dots, v_r such that the angle formed by any two different vectors of the base is at least $\pi/2$, and the matrix $-M$ is the matrix of scalar products of pairs of vectors of the basis. Therefore the inverse matrix $-M^{-1}$ is the matrix of scalar products of pairs of vectors of a basis of vectors such that the angle formed by any two of the vectors is at most $\pi/2$. This implies that all the entries of $-M^{-1}$ are non-negative. \square

Hence, if we require in (4) that each b_i and each a_i are non-negative integers, then we get that b_0 has to be equal to 0 or to 1, and in this last case we get that $b_1 = \dots = b_r = 0$. Hence, we have the following immediate consequence:

Corollary 6. *If α is a wedge realizing an adjacency from E_j to E_0 (with $j \neq 0$), and (b_0, \dots, b_r) are the intersection numbers $Y_s \cdot E_i$ associated with the generic member of a good wedge representative as in (4), then b_0 is equal to 0. Moreover a_0 is positive, that is the divisor E_0 appears in the support of Y_0 .*

Proof. Since α realizes an adjacency from E_j to E_0 we have $b_j \neq 0$. Then $b_0 = 0$. Now in the first row of system (4) in order to have the equality $b_0 = 0$ we need that $\sum_{j=0}^r a_j k_{0,j} = -1$. By definition all a_j and all $k_{0,j}$ except $k_{0,0}$ are non-negative. This implies that a_0 is different from 0. \square

3.3. The equality (4) can be viewed as a linear system whose indeterminates are a_0, \dots, a_r . It can be used to prove that wedges realizing certain adjacencies with certain prescribed returns do not exist (we are using the terminology of [16]). The method is as follows: the adjacencies and the prescribed returns determine b_0, b_1, \dots, b_r . The existence of the wedge is impossible if the solution of the linear system has either a negative or a non-integral entry.

Using this method it is possible to prove the bijectivity of Nash mapping for many singularities (toric, dihedral...), but it does not suffice for all of them. It is interesting to compare this method with the methods of [16] for the E_8 singularity. The set of adjacencies with prescribed returns which this method is not able to

rule coincide precisely with the list of 25 adjacencies with prescribed returns that the second author is not able to rule out only with intersection multiplicity methods.

4. EULER CHARACTERISTIC ESTIMATES

Let \tilde{X} be a compact domain with smooth boundary in a compact complex surface. Let

$$Y_0 = \sum_{i=0}^m c_i Z_i + \sum_{i=0}^r a_i E_i$$

be a divisor in \tilde{X} , where the E_i 's are compact prime divisors contained in the interior of \tilde{X} , and the Z_i 's are prime divisors meeting transversely the boundary of \tilde{X} . We consider a deformation Y_s of the divisor Y_0 such that Y_s is reduced and transversal to the boundary $\partial\tilde{X}$ for $s \neq 0$. Let

$$n : \mathcal{U}_s \rightarrow Y_s$$

be the normalization of Y_s . In this section we bound the Euler characteristic of the normalization \mathcal{U}_s in terms of the topology of the reduced divisor associated with Y_0 , the multiplicities c_i and a_i and the number of intersection points of Y_0 with Y_s , for $s \neq 0$.

We do first the case when Y_0 is a normal crossing divisor. We denote by $(Y_0)^{red}$ the reduced divisor associated with Y_0 .

4.1. Local normal crossings case. In this case \tilde{X} is a ball B_ϵ centred at the origin of \mathbb{C}^2 , and Y_0 is defined by $f_0 = x^a y^b = 0$, where x and y are the coordinates of \mathbb{C}^2 . The divisor Y_s is defined by $f_s = 0$, where f_s is a 1-parameter holomorphic deformation of f_0 such that f_s is reduced for $s \neq 0$. We have the following bound:

Lemma 7. *If s is small enough then the Euler characteristic of the normalization \mathcal{U}_s of Y_s satisfies:*

$$(5) \quad \chi(\mathcal{U}_s) \leq \sum_{p \in Y_s \cap Y_0} I_p((Y_0)^{red}, Y_s).$$

Proof. The only connected orientable surface with boundary which has positive Euler characteristic is the disc. Hence $\chi(\mathcal{U}_s)$ is bounded above by the number of connected components of \mathcal{U}_s which are discs.

Let W_s be an irreducible component of Y_s whose normalization is a disc. Its boundary $W_s \cap \mathbb{S}_\epsilon$ is a circle which deforms to one of the components of $Y_0 \cap \mathbb{S}_\epsilon$, that is either to $V(x) \cap \mathbb{S}_\epsilon$ or to $V(y) \cap \mathbb{S}_\epsilon$. Both cases are symmetric. In the first case the equation g_s of W_s degenerates to x^c for a certain $c \leq a$, that is $g_0 = x^c$. Thus the circle $W_s \cap \mathbb{S}_\epsilon$ loops c times around the $V(y)$, and hence represents a non-trivial element in $\pi_1(B_\epsilon \setminus V(y))$. The normalization of the component W_s is a mapping from a disc to W_s . If W_s does not meet $V(y)$ the circle $W_s \cap \mathbb{S}_\epsilon$ would be a trivial element in $\pi_1(B_\epsilon \setminus V(y))$, and this is not the case.

We conclude that each component of Y_s whose normalization is a disc has at least one intersection point with the union of the axis. This proves the lemma. \square

4.2. Global normal crossings case. We assume Y_0 to be a normal crossings divisor. Define

$$\begin{aligned}\dot{E}_i &= E_i \setminus \text{Sing}((Y_0)^{\text{red}}), \\ \dot{Z}_i &= Z_i \setminus \text{Sing}((Y_0)^{\text{red}}),\end{aligned}$$

for any i .

Lemma 8. *If s is small enough then the Euler characteristic of the normalization \mathcal{U}_s of Y_s satisfies*

$$(6) \quad \chi(\mathcal{U}_s) \leq \sum_{i=0}^m c_i \chi(\dot{Z}_i) + \sum_{i=0}^r a_i \chi(\dot{E}_i) + \sum_{p \in Y_s \cap Y_0} I_p((Y_0)^{\text{red}}, Y_s).$$

Proof. Choose small balls B_1, \dots, B_k inside \tilde{X} centred in each of the singular points of $(Y_0)^{\text{red}}$. Choose tubular neighbourhoods T_i (respectively T'_i) around each component Z_i (respectively E_i), which are so small that their boundaries meet the boundary of each of the balls B_j transversely. Define

$$\begin{aligned}U_i &:= T_i \setminus \cup_{j=1}^k B_j, \\ W_i &:= T'_i \setminus \cup_{j=1}^k B_j.\end{aligned}$$

We have product structures

$$U_i \cong (Z_i \setminus \cup_{j=1}^k B_j) \times D$$

and

$$W_i \cong (E_i \setminus \cup_{j=1}^k B_j) \times D.$$

Let

$$\begin{aligned}\rho_i &: U_i \rightarrow Z_i \setminus \cup_{j=1}^k B_j, \\ \kappa_i &: W_i \rightarrow E_i \setminus \cup_{j=1}^k B_j\end{aligned}$$

be the projections onto the first factors.

If s is small enough the compositions

$$\rho_i \circ n|_{n^{-1}(Y_t \cap U_i)} n^{-1}(Y_t \cap U_i) \rightarrow Z_i \setminus \cup_{j=1}^k B_j$$

$$\kappa_i \circ n|_{n^{-1}(Y_t \cap W_i)} n^{-1}(Y_t \cap W_i) \rightarrow E_i \setminus \cup_{j=1}^k B_j$$

are branched covers of degree c_i and a_i for any i . By Hurwitz formula we find

$$\chi(n^{-1}(Y_t \cap U_i)) \leq c_i \chi(\dot{Z}_i)$$

and

$$\chi(n^{-1}(Y_t \cap W_i)) \leq a_i \chi(\dot{E}_i).$$

Since $n^{-1}(Y_s \cap \partial B_i)$ is a union of circles for any i and the Euler characteristic of a circle is equal to 0, we have

$$\chi(\mathcal{U}_s) = \sum_{i=1}^k \chi(n^{-1}(Y_t \cap B_i)) + \sum_{i=0}^m \chi(n^{-1}(Y_t \cap U_i)) + \sum_{i=0}^r \chi(n^{-1}(Y_t \cap W_i)).$$

Using the local bound obtained in paragraph 4.1 and the bounds above we get the required bound. \square

4.3. General local case. In this case Y_0 is defined by $f_0 = \prod_{i=0}^m g_i^{c_i} = 0$, where the g_i are irreducible and reduced analytic function germs. We denote by μ_i the Milnor number of g_i at the origin. We take a Milnor ball B_ϵ for f_0 as the space \tilde{X} . The divisor Y_s is defined by $f_s = 0$, where f_s is a 1-parameter holomorphic deformation of f_0 such that f_s is reduced for $s \neq 0$. We consider a sufficiently small δ so that $f_0^{-1}(\delta) \cap B_\epsilon$ is the Milnor fibre of f_0 . We will use and generalize in certain sense the following equality that was proved in [13]. We start by giving an alternative proof of the equality.

Lemma 9 ([13]). *The Euler characteristic of the Milnor fibre of f_0 is equal to:*

$$(7) \quad \chi(f_0^{-1}(\delta) \cap B_\epsilon) = \sum_{i=0}^m c_i (1 - \mu_i - I_O(g_i, \prod_{j \neq i} g_j)).$$

Proof. Given a vector v of \mathbb{C}^2 we denote by τ_v the translation of \mathbb{C}^2 associated with v . We choose m vectors v_1, \dots, v_m in \mathbb{C}^2 such that for any t small enough and $i \neq j$ the curves $V(g_i \circ \tau_{tv_i} - t)$ and $V(\prod_{j \neq i} g_j \circ \tau_{tv_j} - t)$ meet transversely in B_ϵ .

Consider the deformation $F_t := \prod_{i=0}^m (g_i \circ \tau_{tv_i} - t)^{c_i}$. An easy local argument shows that for small enough t and any $s \in D_\delta \setminus \{0\}$ the set $F_t^{-1}(s)$ is smooth at the meeting points with ∂B_ϵ and transverse to it. This implies the existence of a finite subset of critical values Δ_t of D_δ such that the restriction

$$F_t : B_\epsilon \cap F_t^{-1}(D_\delta \setminus \Delta_t) \rightarrow D_\delta \setminus \Delta_t$$

is a locally trivial fibration with fibre diffeomorphic to the Milnor fibre of f_0 . See Theorem 2.2 of [2] for a proof of these facts in a much more general context.

Fix a small enough t different from 0. We view $F_t^{-1}(s)$ as a deformation of the normal crossings divisor $F_t^{-1}(0)$ inside B_ϵ and study it like in the global normal crossings case. The irreducible components of this divisor are $Z_i = V(g_i \circ \tau_{tv_i} - t)$, for $i = 0, \dots, m$. The component Z_i is a translation of the Milnor fibre of g_i , and, hence, its Euler characteristic is equal to $1 - \mu_i$. Consequently, using that the curve Z_i meets transversely the union $\cup_{j \neq i} Z_j$ and the conservativity of intersection multiplicity, we obtain

$$\chi(\dot{Z}_i) = 1 - \mu_i - I_O(g_i, \prod_{j \neq i} g_j).$$

Observe that the piece of the Milnor fibre contained in a neighbourhood of a singularity of $F_t^{-1}(0)$ is a union of cylinders because locally $F_t^{-1}(0)$ is normal crossings. Decomposing the Milnor fibre as in Lemma 8 and adding the Euler characteristics of the corresponding pieces we obtain the equality. \square

After this Lemma we can prove the Euler characteristic bound that we want:

Proposition 10. *If s is small enough we have*

$$(8) \quad \chi(\mathcal{U}_s) \leq \sum_{i=0}^m c_i (1 - \mu_i - I_O(g_i, \prod_{j \neq i} g_j)) + \sum_{p \in Y_s \cap Y_0} I_p((Y_0)^{red}, Y_s).$$

Proof.

A PARTICULAR CASE: the divisor Y_s does not meet the origin for $s \neq 0$.

In order to reduce the problem to the global normal crossings case we consider the minimal embedded resolution

$$\pi : \tilde{X} \rightarrow B_\epsilon$$

of $V(f_0)$. Let $\{E_i\}_{i=1}^r$ be the irreducible components of the exceptional divisor. For any $s \in D_\delta$ we denote by V_s the pullback of Y_s by π . Since the divisor Y_s does not meet the origin when $s \neq 0$ we have that it is isomorphic to V_s and that V_s does not meet the exceptional divisor of π . Then it is enough to prove the bound for the Euler characteristic of the divisor V_s for $s \neq 0$.

The divisor V_0 decomposes as $V_0 = \sum_{i=0}^m c_i Z_i + \sum_{i=0}^r a_i E_i$, where the c_i 's depend only on the resolution of singularities of $f_0^{-1}(0)$ and the a_i 's are deduced from the c_i 's solving the linear system derived from the identities $V_0 \cdot E_i = 0$ (notice that V_s does not meet any E_i because Y_s does not meet the origin and then $V_s \cdot E_i = 0$ for all i).

Using the bound obtained in paragraph 4.1 and the fact that \dot{Z}_i is a punctured disc for any i we obtain

$$(9) \quad \chi(\mathcal{U}_s) \leq \sum_{i=0}^r a_i \chi(\dot{E}_i) + \sum_{p \in V_0 \cap V_s} I_p(V_s, (V_0)^{red}).$$

Using the fact that the number of intersection points of Y_s and $(Y_0)^{red}$ counted with multiplicity coincides with the number of intersection points of V_s and $(V_0)^{red}$ counted with multiplicity, after Lemma 9, in order to prove the proposition it only rest to check that the first sum of the right side of (9) coincides with the Euler characteristic of the Milnor fibre of f_0 .

For this we observe that the divisor $V_0 = \sum_{i=0}^m c_i Z_i + \sum_{i=0}^r a_i E_i$ is equal to the total transform of $V(f_0)$ by the modification π . This is because the coefficients a_i are also characterized by the equalities $V_0 \cdot E_i = 0$ for any i . The Euler characteristic of the Milnor fibre is given then by

$$(10) \quad \chi(f_0^{-1}(s)) = \sum_{i=0}^r a_i \chi(\dot{E}_i).$$

Indeed, if W_s is the pullback of the Milnor fibre $f_0^{-1}(s)$ by π we apply to W_s the procedure of the proof of Lemma 8 and the following easy facts:

- The piece of Milnor fibre contained at the balls neighbouring singular points of the total transform is a union of cylinders.
- The coverings associated to the part of Milnor fibre contained at the tubular neighbourhoods of \dot{E}_i and \dot{Z}_i are unramified.
- Each set \dot{Z}_i is a punctured disc.

GENERAL CASE. We reduce the proof to the previous particular case by a deformation argument. Recall that τ_v denotes the translation in the direction of a vector v . Let v_t be a holomorphic family of vectors in \mathbb{C}^2 with $v_0 = O$ and such that for t small enough $V(f_0 \circ \tau_{v_t})$ does not meet the origin. It is easy to check that the 2-parameter family $F_{t,s} := f_s \circ \tau_{v_t}$ has the following properties:

- (i) The set of parameters Δ such that $V(F_{t,s})$ meets the origin is a proper closed analytic subset in the parameter space.
- (ii) There exist positive $\eta < \delta$ such that for any s with $0 < |s| \leq \delta$ and any t satisfying $0 \leq |t| < \eta$ the normalization of $V(F_{t,s}) \cap B_\epsilon$ is diffeomorphic to the normalization of $V(F_{0,s}) = V(f_s)$.

Choose a parametrized curve in the parameter space of the family of the form $(t(s), s)$ with $t(0) = 0$ and such that for $s \neq 0$ small enough $t(s)$ is non-zero and

avoids Δ . Then, the normalization of $V(F_{t(s),s})$ is diffeomorphic to the normalization of $V(f_s)$ for any s . Applying the particular case to the family $V(F_{t(s),s})$ we prove the proposition for the general case. \square

4.4. General global case. For any component E_i we consider the set of irreducible components of the germ of E_i at each point of $Sing((Y_0)^{red})$. We denote these germs by $\{(\Gamma_k, p_k)\}_{k=1}^d$. We denote by μ_{E_i} the sum of Milnor numbers of these local branches, by ν_{E_i} the number of branches and define

$$\eta_{E_i} := \sum_{k=1}^d \sum_{l \neq k} I_{p_k}(\Gamma_k, \Gamma_l).$$

We also define the analogous numbers μ_{Z_i} , ν_{Z_i} and η_{Z_i} for any divisor Z_i .

For any i we denote by \dot{E}_i (respectively \dot{Z}_i) the set $E_i \setminus Sing((Y_0)^{red})$ (respectively $Z_i \setminus Sing((Y_0)^{red})$).

Proposition 11. *For non-zero and small enough s we have*

$$(11) \quad \chi(\mathcal{U}_s) \leq \sum_{i=0}^m c_i(\chi(\dot{Z}_i) + \theta_{Z_i}) + \sum_{i=0}^r a_i(\chi(\dot{E}_i) + \theta_{E_i}) + \sum_{p \in Y_s \cap Y_0} I_p(Y_s, (Y_0)^{red}),$$

where $\theta(Z_i)$ and $\theta(E_i)$ are defined by

$$\begin{aligned} \theta_{Z_i} &:= \nu_{Z_i} - \mu_{Z_i} - \eta_{E_i} - Z_i \cdot ((Y_0)^{red} - Z_i), \\ \theta_{E_i} &:= \nu_{E_i} - \mu_{E_i} - \eta_{Z_i} - E_i \cdot ((Y_0)^{red} - E_i). \end{aligned}$$

Proof. The proof follows the scheme of the proof of Lemma 8. We consider small Milnor balls around the singular points of $(Y_0)^{red}$ and small tubular neighbourhoods around the connected components of the complement of these balls in $(Y_0)^{red}$. We split \mathcal{U}_s into pieces, each being the part that maps into one of the neighbourhoods just defined. We bound the Euler characteristic of the parts corresponding to tubular neighbourhoods using Hurwitz formula as in the proof of Lemma 8. We bound the Euler characteristic of the pieces corresponding to the Milnor balls using Proposition 10. Summing up the contributions and rearranging terms we get the desired expression. \square

5. BIJECTIVITY OF THE NASH MAP FOR NORMAL SURFACE SINGULARITIES

Theorem 12. *Nash mapping is bijective for any normal surface singularity defined over an algebraically closed field of characteristic equal to 0.*

Proof. The argument in paragraph 2.3 shows that it is enough to deal with the complex case.

Let (X, O) be a complex normal surface singularity. If Nash mapping is not bijective then, by Theorem 2 there exists a wedge α realizing an adjacency from a component E_j of the exceptional divisor of the minimal resolution to a different component E_0 . We take a good representative $\alpha|_{\mathcal{U}}$ and define the divisors Y_0 and Y_s as in paragraph 3.1. As explained there, since \mathcal{U}_s is a disc, the lifting

$$\tilde{\alpha}_s : \mathcal{U}_s \rightarrow \tilde{X}$$

is the normalization of Y_s . We will use estimates of Section 4 to get a contradiction with the fact that the Euler characteristic of \mathcal{U}_s is 1. In this way we show the non-existence of α and, by Theorem 2, that the Nash mapping is bijective for normal surface singularities.

5.1. We are going to improve slightly the estimate for $\chi(\mathcal{U}_s)$ given in Proposition 11. Remember that since α realizes an adjacency from E_j to E_0 with $j \neq 0$, the divisor E_0 appears in Y_0 (see Corollary 6). Besides, we have a single Z_i , that is Z_0 in (2), which has the topology of a disc and intersects transversely E_0 at a smooth point of E . Moreover the divisor Y_0 is reduced at the generic point of Z_0 . We split \mathcal{U}_s in two pieces as follows. Let \tilde{X}_1 be a small compact tubular neighbourhood around the disc Z_0 in \tilde{X} . Define \tilde{X}_2 as the closure of the complement of \tilde{X}_1 in \tilde{X} . For s non-zero and small enough the divisor Y_s meets transversely the boundaries of \tilde{X}_1 and \tilde{X}_2 . For $i = 1, 2$ define \mathcal{U}_s^i as the normalization of $Y_s \cap \tilde{X}_i$. Since the intersection $\mathcal{U}_s^1 \cap \mathcal{U}_s^2$ is a union of circles we have that

$$(12) \quad \chi(\mathcal{U}_s) = \chi(\mathcal{U}_s^1) + \chi(\mathcal{U}_s^2).$$

Let us give an improved bound for the Euler characteristic of \mathcal{U}_s^1 using the methods of paragraph 4.1. Let B be a Milnor ball for Y_0 around the point $p = E_0 \cap Z_0$. We may choose local coordinates (x, y) around p so that we have $E_0 = V(y)$ and $Z_0 = V(x)$. Let g_s be the family of functions defining the divisor Y_s locally around p . We have, up to a unit, the equality $g_0 = xy^{a_0}$. The Euler characteristic of \mathcal{U}_s^1 is bounded by the number of topological discs in the normalization of $V(g_t) \cap B$. In principle the number of circles in $\partial B \cap Y_t$ is at most $a_0 + 1$. There certainly appears one circle K_s which is a small deformation of $V(x) \cap \partial B$. By the connectivity of \mathcal{U}_s , the boundary of the connected component of \mathcal{U}_s^1 containing K_s can not consist only of K_s . This implies that the maximal number of discs that can appear in \mathcal{U}_s^1 is $a_0 - 1$ and hence

$$(13) \quad \chi(\mathcal{U}_s^1) \leq a_0 - 1.$$

The Euler characteristic of \mathcal{U}_s^2 is bounded using Proposition 11. Notice the following identities,

$$\nu_{E_0 \cap \tilde{X}_2} = \nu_{E_0} - 1,$$

$$\mu_{E_0 \cap \tilde{X}_2} = \mu_{E_0},$$

$$(E_0 \cap \tilde{X}_2) \cdot ((Y_0)^{red} \cap \tilde{X}_2 - E_0 \cap \tilde{X}_2) = E_0 \cdot ((Y_0)^{red} - E_0) - 1,$$

which imply that

$$\theta_{E_0 \cap \tilde{X}_2} = \theta_{E_0}.$$

Then, by (12) we obtain

$$(14) \quad \chi(\mathcal{U}_s) \leq a_0 - 1 + \sum_{i=0}^r a_i (\chi(\dot{E}_i) + \theta_{E_i}) + \sum_{p \in Y_s \cap Y_0 \cap \tilde{X}_2} I_p(Y_s, (Y_0)^{red}).$$

Note that the last term is the total number of *returns*. Defining $\delta_{a_j} = 1$ if $a_j \neq 0$ and $\delta_{a_j} = 0$ if $a_j = 0$ we have the obvious bound

$$(15) \quad \sum_{p \in Y_s \cap Y_0 \cap \tilde{X}_2} I_p(Y_s, (Y_0)^{red}) \leq \sum_{j=0}^r \delta_{a_j} b_j.$$

If we denote by $k_{i,j}$ the intersection product $E_i \cdot E_j$, by Equation (4) we have that

$$\sum_{j=0}^r b_j = \sum_{j=0}^r \sum_{i=0}^r \delta_{a_j} a_i k_{i,j} + 1.$$

Regrouping and coming back to (15) we get the following

$$(16) \quad \sum_{p \in Y_s \cap Y_0 \cap \tilde{X}_2} I_p(Y_s, (Y_0)^{red}) \leq \sum_{i=0}^r a_i \left(\sum_{j=0}^r \delta_{a_j} k_{i,j} \right) + 1.$$

Now, in one hand, denoting by g_i the genus of the normalization of E_i , we have

$$(17) \quad \chi(\dot{E}_i) = 2 - 2g_i - \nu_{E_i}.$$

On the other hand we have that

$$\begin{aligned} E_0 \cdot ((Y_0)^{red} - E_0) &= \sum_{j \neq 0} \delta_{a_j} k_{0,j} + E_0 \cdot Z_0 = \sum_{j \neq 0} \delta_{a_j} k_{0,j} + 1, \\ E_i \cdot ((Y_0)^{red} - E_0) &= \sum_{j \neq i} \delta_{a_j} k_{i,j} \quad \text{for any } 1 \leq i \leq r. \end{aligned}$$

and hence

$$(18) \quad \theta_{E_0} = \nu_{E_0} - \mu_{E_0} - \eta_{E_0} - \sum_{j \neq 0} \delta_{a_j} k_{0,j} - 1,$$

$$(19) \quad \theta_{E_i} = \nu_{E_i} - \mu_{E_i} - \eta_{E_i} - \sum_{j \neq i} \delta_{a_j} k_{i,j} \quad \text{for any } 1 \leq i \leq r.$$

Performing substitutions (17)-(19) in (14) and using (16), we get to the following:

$$(20) \quad \chi(\mathcal{U}_s) \leq \sum_{i=0}^r a_i (2 - 2g_i - \mu_{E_i} - \eta_{E_i} + k_{i,i}).$$

By negative definiteness, for any $0 \leq i \leq r$, the self-intersection $k_{i,i}$ is a negative integer. Observe that, since $\pi : \tilde{X} \rightarrow X$ is the minimal resolution, for any $0 \leq i \leq r$, if $k_{i,i}$ is equal to -1 , then either the divisor E_i is singular or it has positive genus (otherwise it is a smooth rational divisor with self-intersection equal to -1 and the resolution is non-minimal). If the divisor E_0 has an irreducible singularity then μ_{E_0} is at least 2. If the divisor E_i has a singular point with several irreducible branches then η_{E_i} is at least 2. Therefore we have

$$a_i (2 - 2g_i - \mu_i - \eta_i + k_{i,i}) \leq 0$$

for any i (note that $a_i \geq 0$). Hence we get that $\chi(\mathcal{U}_s) \leq 0$. This is a contradiction because we know that \mathcal{U}_s is a disc. \square

6. THE NON-NORMAL CASE

Consider a Hironaka resolution of singularities of an algebraic variety (a resolution which is an isomorphism outside the singular locus). With any divisorial component C of the exceptional locus we associate the set N_C consisting of arcs in the variety centred at the singular locus, not contained in it and whose lifting to the resolution is centred at C . It is an irreducible Zariski closed subset of the space of arcs in the variety which are not contained in the singular set. Its scheme theoretical generic point γ_C is a formal arc in the variety defined over the residue field of N_C .

Given any field K , a *formal K -wedge* in X is a morphism

$$\alpha : \text{Spec}(K[[t, s]]) \rightarrow X.$$

It can be seen as a family of arcs parametrized by s . Its associated special arc is the restriction of α to the $\{s = 0\}$ locus of $\text{Spec}(K[[t, s]])$.

We will use the following theorem:

Theorem 13 (Theorem 5.1 [25]). *An essential component C of the exceptional divisor of a resolution of singularities of a variety is in the image of the Nash map if and only if any K -wedge in the variety whose special arc equals the generic point γ_C lifts to the resolution.*

Let X_1 be any reduced algebraic surface defined over a field of characteristic equal to 0. Let

$$n : X_2 \rightarrow X_1$$

be the normalization and

$$\pi : X_3 \rightarrow X_2$$

be the minimal resolution of the singularities of X_2 .

Let $\cup_{i=1}^r E_i$ be a decomposition into irreducible components of the exceptional divisor of π . By the minimality of the resolution all these components are essential.

Let $n^{-1}(\text{Sing}(X_1)) = \cup_{i=1}^s A_i$ be a decomposition into irreducible components of the preimage of the singular set of X_1 by the normalization. Denote by B_i the strict transform of A_i by π . The decomposition into irreducible components of the exceptional divisor of the resolution $n \circ \pi$ is given by

$$(\cup_{i=1}^s B_i) \cup (\cup_{i=1}^r E_i).$$

All these components are essential.

Any K -wedge whose special arc equals the generic point either of N_{E_i} or of N_{B_i} is a dominant morphism

$$\alpha_1 : \text{Spec}(K[[t, s]]) \rightarrow X_1.$$

Since $\text{Spec}(K[[t, s]])$ is normal, by the universal property of the normalization, it admits a lifting

$$\alpha_2 : \text{Spec}(K[[t, s]]) \rightarrow X_2.$$

Assume that the special arc of α_1 is the generic point of a certain N_{B_j} for a certain j . Let η_j be the generic point of A_j in the scheme X_2 . Then the image by α_2 of the closed point of $\text{Spec}(K[[t, s]])$ is η_j . Since π is an isomorphism in a neighbourhood of the preimage $\pi^{-1}(\eta_j)$, the wedge α_2 lifts to X_3 .

Assume that the special arc of α_1 is the generic point of N_{E_i} for a certain i . Then α_2 is a wedge in X_2 whose special arc is the generic point of N_{E_i} . Since E_i is an essential component of the exceptional divisor of π , and X_2 is normal, case for which we have just proved that Nash mapping is bijective, we have that E_i is at the image of the Nash map for X_2 . This implies by Theorem 13 that α_2 admits a lifting to X_3 .

We have proved that any K -wedge in X_1 whose special arc equals the generic point of an essential component of $n \circ \pi$ lifts to X_3 . This shows by Theorem 13 the bijectivity of the Nash map for X_1 .

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